CROSSFIRE: PIRATE'S GUNNERY MANUAL

Arrrrr! If ye wish to survive on the high seas, then ye'll need to know how to handle yer cannon. To whit, ye must master the art and science of gunnery. As any wench can tell ye, t'art comes wi' practice; the science begins and ends with numerology.

There be two possibilities: ye be at a certain elevation h and wish to fire at angle θ and velocity v to cover range d *(fig. 1)*, such as a fort wishing to sink a vessel, or ye be at sea level at range d and wish to fire at angle θ and velocity v to reach elevation h *(fig. 2),* such as a corsair wishing to bombard a rich town. Pirates prefer yon latter scenario. An honest pirate wishin' to attack a fat galleon is a special case of either, ye bilge rats.

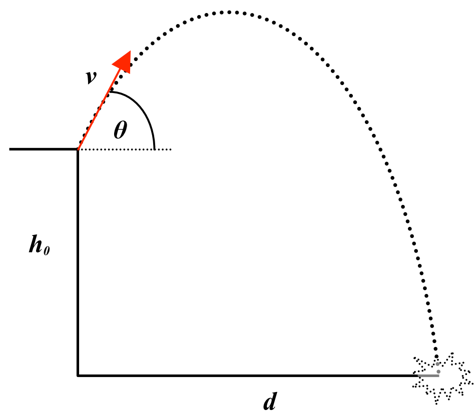

Fig. 1

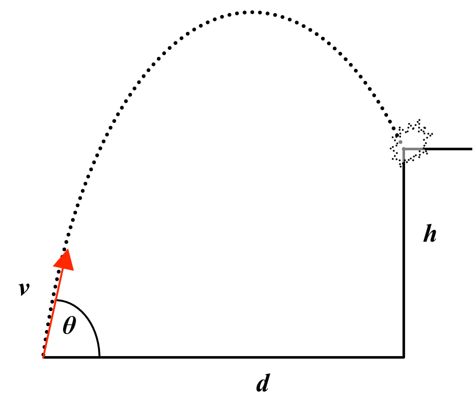

Fig. 2

For both cases, the height *h* reached after time *t* is given by

$$h = \tfrac{1}{2}at^2 + (v\sin\theta)t + h_0$$

Also, the range d can be found by

$$d = (v\cos\theta)t$$

Which can be rearranged and substituted into the first equation for t

$$h = \tfrac{1}{2}a\left(\frac{d}{v\cos\theta}\right)^2 + (v\sin\theta)\left(\frac{d}{v\cos\theta}\right) + h_0$$

or

$$h = \tfrac{1}{2}a\left(\frac{d}{v\cos\theta}\right)^2 + d(\tan\theta) + h_0$$

If ye take $a = -9.8\text{m/s}^2 = -g$, then the general formula for finding h_0 or h (assuming the other is zero) is

$$h - h_0 = d(\tan\theta) - \tfrac{1}{2}g\left(\frac{d}{v\cos\theta}\right)^2$$

which becomes

$$\boxed{h_0 = \tfrac{1}{2}g\left(\frac{d}{v\cos\theta}\right)^2 - d(\tan\theta)} \text{ or } \boxed{h = d(\tan\theta) - \tfrac{1}{2}g\left(\frac{d}{v\cos\theta}\right)^2} \text{ respectively}$$

If a projectile be fired with a muzzle velocity v_0 and angle θ to land at the same level, the range of the projectile is:

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

If a projectile be fired wi' a muzzle velocity v_0 and angle θ, and lands at the same level, the maximum height of the projectile is:

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$

If yer captain tells ye the initial muzzle velocity and firing angle, a cabin boy can solve for height or range. But what if ye ken only the relative elevation and muzzle velocity? That is when a master gunner knows how to aim his cannon!

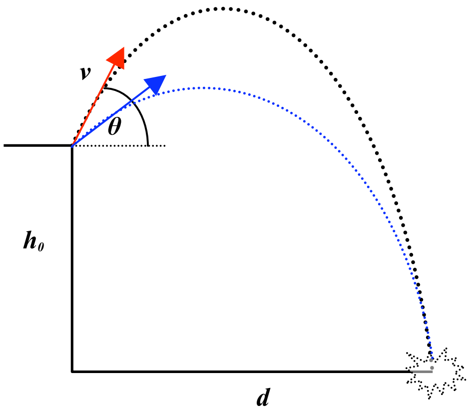

Fig. 1

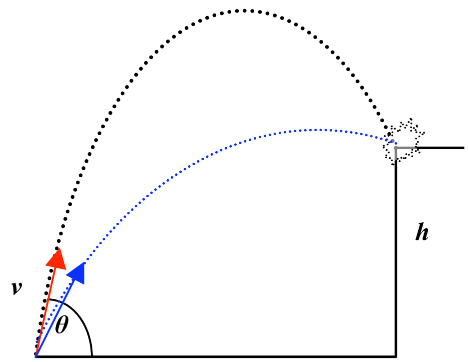

Fig. 2

This canna be so easily solved, because the angle θ is part of two different trigonometric functions. To proceed, ye must either choose the algebraic approach or the trigonometric approach or the other trigonometric approach: all involve quadratic solutions, because two angles will satisfy the conditions, as shown above.

A tip o' the cap to gunner's mates Andrew Rothfuss, Haoyang Liu and Jasper Chen for their assistance in scrivening the following:

Algebraic approach

Firstly, call $h-h_0=y$, $d=x$, $v\sin\theta=v_y$ and call $v\cos\theta=v_x$; then the previous general-purpose solution becomes

$$y = x\left(\frac{v_y}{v_x}\right) - \tfrac{1}{2}g\left(\frac{x}{v_x}\right)^2$$

It also be known that $v^2 = v_y^2 + v_x^2$; solving for v_x and then substituting $\sqrt{v^2 - v_y^2} = v_x$ be harder than swilling bad ale: it be more lucrative to solve for v_y and substitute $\sqrt{v^2 - v_x^2} = v_y$

$$yv_x = xv_y - \frac{\tfrac{1}{2}gx^2}{v_x}$$

$$v_y = \left(\frac{y}{x}\right)v_x + \frac{\tfrac{1}{2}gx}{v_x}$$

$$\sqrt{v^2 - v_x^2} = \left(\frac{y}{x}\right)v_x + \frac{\tfrac{1}{2}gx}{v_x}$$

$$v^2 - v_x^2 = \left[\left(\frac{y}{x}\right)v_x + \frac{\tfrac{1}{2}gx}{v_x}\right]^2$$

$$v^2 - v_x^2 = \left[\left(\frac{y}{x}\right)v_x\right]^2 + 2\left(\frac{y}{x}\right)\left(\tfrac{1}{2}gx\right) + \left[\frac{\left(\tfrac{1}{2}gx\right)}{v_x}\right]^2$$

$$\left(\frac{y}{x}\right)^2 v_x^2 + 2\left(\frac{y}{x}\right)\left(\tfrac{1}{2}gx\right) + \left[\frac{\left(\tfrac{1}{2}gx\right)}{v_x}\right]^2 - v^2 + v_x^2 = 0$$

$$\left[\left(\frac{y}{x}\right)^2 + 1\right]v_x^2 + 2\left(\frac{y}{x}\right)\left(\tfrac{1}{2}gx\right) - v^2 + \left[\frac{\left(\tfrac{1}{2}gx\right)}{v_x}\right]^2 = 0$$

$$\left[\left(\frac{y}{x}\right)^2 + 1\right]\left(v_x^2\right)^2 + \left[2\left(\frac{y}{x}\right)\left(\tfrac{1}{2}gx\right) - v^2\right]v_x^2 + \left(\tfrac{1}{2}gx\right)^2 = 0$$

$$\left[\left(\frac{y}{x}\right)^2 + 1\right]\left(v_x^2\right)^2 + \left(yg - v^2\right)v_x^2 + \left(\tfrac{1}{2}gx\right)^2 = 0$$

Use the quadratic equation to solve for v_x^2, where

$$a = \left(\frac{y}{x}\right)^2 + 1$$
$$b = yg - v^2$$
$$c = \left(\tfrac{1}{2}gx\right)^2$$

and take the positive square roots (For those souls of a philoshophic bent, the negative roots be the problem in reverse lookin' into th' past) and then solve for the angle $\theta=\cos^{-1}(v_x/v)$

Trigonometric approach
Start with the general-purpose formula

$$h - h_0 = d(\tan\theta) - \tfrac{1}{2} g\left(\frac{d}{v\cos\theta}\right)^2$$

$$h - h_0 = d(\tan\theta) - \tfrac{1}{2} g\left(\frac{d}{v}\right)^2\left(\frac{1}{\cos\theta}\right)^2$$

$$h - h_0 = d(\tan\theta) - \tfrac{1}{2} g\left(\frac{d}{v}\right)^2 \sec^2\theta$$

$$h - h_0 = d(\tan\theta) - \tfrac{1}{2} g\left(\frac{d}{v}\right)^2 \left(\tan^2\theta + 1\right)$$

$$h - h_0 = d(\tan\theta) - \left[\tfrac{1}{2} g\left(\frac{d}{v}\right)^2\right]\left(\tan^2\theta\right) - \left[\tfrac{1}{2} g\left(\frac{d}{v}\right)^2\right]$$

$$\left[\tfrac{1}{2} g\left(\frac{d}{v}\right)^2\right]\left(\tan^2\theta\right) - d(\tan\theta) + \left[\tfrac{1}{2} g\left(\frac{d}{v}\right)^2\right] + (h - h_0) = 0$$

Use the quadratic equation to solve for tanθ, where

$$a = \tfrac{1}{2} g\left(\frac{d}{v}\right)^2$$

$$b = -d$$

$$c = \tfrac{1}{2} g\left(\frac{d}{v}\right)^2 + (h - h_0)$$

and either h or h_0 equals zero; solve for angle $\theta = \tan^{-1}$

The other trigonometric approach
Start with the general-purpose formulae

$$(h - h_0) = -\frac{1}{2}gt^2 + vt\sin\theta \rightarrow \sin\theta = \frac{(h - h_0) + \frac{1}{2}gt^2}{vt}$$

$$d = vt\cos\theta \rightarrow \cos\theta = \frac{d}{vt}$$

Then use the first trig identity:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{(h - h_0) + \frac{1}{2}gt^2}{vt}\right)^2 + \left(\frac{d}{vt}\right)^2 = 1$$

$$\frac{(h - h_0)^2 + gt^2(h - h_0) + \frac{1}{4}g^2t^4}{vt} + \frac{d^2}{v^2t^2} = 1$$

$$\frac{1}{v^2t^2}\left(\frac{1}{4}g^2t^4 + g(h - h_0)t^2 + (h - h_0)^2 + d^2\right) - 1 = 0$$

$$\frac{1}{v^2t^2}\left(\frac{1}{4}g^2(t^2)^2 + (g(h - h_0) - v^2)(t^2) + (h - h_0)^2 + d^2\right) = 0$$

$$\frac{1}{4}g^2(t^2)^2 + (g(h - h_0) - v^2)(t^2) + (h - h_0)^2 + d^2 = 0$$

Use the quadratic equation to solve for t^2, where

$$a = \frac{1}{4}g^2$$
$$b = g(h - h_0) - v^2$$
$$c = (h - h_0)^2 + d^2$$

t is the time in seconds when the projectile reaches the given distance and height.

Solve for θ using $\theta = \arccos\frac{d}{vt}$ where $t = \sqrt{t^2}$ using the positive values of t.